

Talk in Peking University

July 5-7, 2010

# Extensions of a theorem of Hsu and Robbins on the convergence rates in the law of large numbers

LIU Quansheng (with HAO Shunli)

Univ. de Bretagne-Sud

# 1 Introduction

## 1.1 Convergence rate in the law of large numbers: the iid case

Consider i.i.d. r.v.  $X_j$  with  $EX_j = 0$ . Let

$$S_n = X_1 + \dots + X_n.$$

Law of Large numbers:

$$\frac{S_n}{n} \rightarrow 0:$$

Question: at what rate  $P(|S_n| > n^\alpha) \rightarrow 0$ ?

## The theorem of Hsu-Robbins-Erdos

Hsu and Robbins (1947):

$$EX_1^2 < \infty \Rightarrow \sum_n P(|S_n| > n^\alpha) < \infty \quad \forall \alpha > 0:$$

("complete convergence", which implies a.s. convergence)

Erdos (1949): the converse also holds:

$$EX_1^2 < \infty \Leftrightarrow \sum_n P(|S_n| > n^\alpha) < \infty \quad \forall \alpha > 0:$$

Spitzer (1956):

$$\sum_n n^{-1} P(|S_n| > n^\alpha) < \infty \quad \forall \alpha > 0 \text{ whenever } EX_1 = 0:$$

Baum and Katz (1965): for  $p > 1$ ;

$$E|X_1|^p < \infty \Leftrightarrow \sum_n n^{p-2} P(|S_n| > n^\alpha) < \infty \quad \forall \alpha > 0;$$

in particular,

$$E|X_1|^p < \infty \Rightarrow P(|S_n| > n^\alpha) = o(n^{-(p-1)})$$

Question: is it valid for martingale differences?

## 1.2 Convergence rates in the law of large numbers: the martingale case

Is the theorem of Baum and Katz (1965) still valid for martingale differences  $(X_j)$ ?

$$\{\emptyset; \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots;$$

$\forall j$ ,  $X_j$  are  $\mathcal{F}_j$  measurable with  $E[X_j | \mathcal{F}_{j-1}] = 0$

( $\Leftrightarrow S_n = X_1 + \dots + X_n$  is a martingale. )

Lesigne and Volney (2001):  $p \geq 2$

$$E|X_1|^p < \infty \Rightarrow P(|S_n| > n^\alpha) = o(n^{-p=2})$$

and the exponent  $p=2$  is the best possible, even for stationary and ergodic sequences of martingale differences.

Therefore **the theorem of Baum and Katz does not hold for martingale differences without additional conditions.**

[ Curiously, Stoica (2007) claimed that the theorem of Baum and Katz still holds for  $p > 2$  in the case of martingale differences without additional assumption. His claim is a contradiction with the conclusion of Lesigne and Volney (2001), and his proof is wrong: **he chose an element in an empty set!** ]

## 1.3 Under what conditions the theorem of Baum and Katz still holds for martingale differences?

Alsmeyer (1990) proved that the theorem of Baum and Katz of order  $p > 1$  still holds for martingale differences  $(X_j)$  if for some  $\alpha \in (1; 2]$  and  $q > (p - 1)/(\alpha - 1)$ ,

$$\sup_{n \geq 1} \left\| \frac{1}{n} \sum_{j=1}^n E[|X_j| \mid \mathcal{F}_{j-1}] \right\|_q < \infty$$

where  $\|\cdot\|_q$  denotes the  $L^q$  norm.

His result is already nice, but:



(a) it does not apply

**Our objective:** extend the theorem of Baum and Katz (1965) to a large class of martingale arrays, in improving Alsmeyer's result for martingales, by establishing a sharp comparison result between

$$P\left(\sum_{j=1}^{\infty} X_{n;j} > \epsilon\right) \text{ and } \sum_{j=1}^{\infty} P(X_{n;j} > \epsilon)$$

for arrays of martingale differences  $\{X_{n;j} : j \geq 1\}$ .

Our result is sharper than the known ones even in the independent (not necessarily identically distributed) case.

## 2. Main results for martingale arrays

For  $n \geq 1$ , let  $\{(X_{nj}; \mathcal{F}_{nj}) : j \geq 1\}$  be a sequence of martingale differences, and write

$$m_n(\cdot) = \sum_{j=1}^{\infty} E[|X_{nj}| | \mathcal{F}_{n;j-1}]; \quad \in (1; 2];$$

$$S_{n;j} = \sum_{i=1}^j X_{ni}; \quad j \geq 1;$$

$$S_{n;\infty} = \sum_{i=1}^{\infty} X_{ni};$$

**Lemma 1** (Law of large numbers) If for some  $\alpha \in (1; 2]$ ,

$$E m_n(\alpha) := \sum_{j=1}^{\infty} E[|X_{nj}|^\alpha] \rightarrow 0;$$

then for all  $\epsilon > 0$ ,

$$P\{\sup_{j \geq 1} |S_{n;j}| > \epsilon\} \rightarrow 0$$

and

$$P\{|S_{n;\infty}| > \epsilon\} \rightarrow 0:$$

We are interested in their convergence rates.

**Theorem 1** Let  $\Phi : \mathbb{N} \mapsto [0; \infty)$ . Suppose that for some  $\alpha \in (1; 2]$ ;  $q \in [1; \infty)$  and  $\alpha_0 \in (0; 1)$ ,

$$Em_n^q(\cdot) \rightarrow 0 \text{ and } \sum_{n=1}^{\infty} \Phi(n) (Em_n^q(\cdot))^{1-\alpha_0} < \infty: \quad (C1)$$

Then the following assertions are all equivalent:

$$\sum_{n=1}^{\infty} \Phi(n) \sum_{j=1}^{\infty} P\{|X_{nj}| > \alpha\} < \infty \quad \forall \alpha > 0; \quad (1)$$

$$\sum_{n=1}^{\infty} \Phi(n) P\{\sup_{j \geq 1} |S_{nj}| > \alpha\} < \infty \quad \forall \alpha > 0; \quad (2)$$

$$\sum_{n=1}^{\infty} \Phi(n) P\{|S_{n;\infty}| > \alpha\} < \infty \quad \forall \alpha > 0: \quad (3)$$

**Remark.** The condition (C1) holds if for some  $r \in \mathbb{R}$  and  $\alpha_1 > 0$ ,

$$\Phi(n) = O(n^r) \text{ and } \|m_n(\cdot)\|_\infty = O(n^{-\alpha_1}): \quad (\text{C1}')$$

In the case where this holds with  $\alpha_1 = 2$ , Ghosal and Chandra (1998) proved that (1) implies (2); our result is sharper because we have the equivalence.

**Theorem 2** Let  $\Phi : \mathbb{N} \mapsto [0; \infty)$  be such that  $\Phi(n) \rightarrow \infty$ . Suppose that for some  $\alpha \in (1; 2]; q \in [1; \infty)$  and  $\beta_0 \in (0; 1)$ ,

$$\Phi(n) (\mathbb{E} m_n^q(\cdot))^{1-\beta_0} = o(1) \quad (\text{resp: } O(1)): \quad (C2) \quad n \rightarrow \infty$$

**3. Consequences for martingales** We now consider the single martingale case

$$S_j = X_1 + \dots + X_j$$

w.r.t. a filtration

$$\{\emptyset; \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$$

By definition,  $E[X_j | \mathcal{F}_{j-1}] = 0$ .

For simplicity, let us only consider the case where

$$\Phi(n) = n^{p-2} \cdot (n);$$

where  $p > 1$ ,  $\cdot$  is a function slowly varying at  $\infty$ :

$$\lim_{x \rightarrow \infty} \frac{\cdot(x)}{\cdot(x)} = 1 \quad \forall \epsilon > 0:$$



Notice that

$$S_{n=n} \rightarrow 0 \text{ a.s. iff } P\left(\sup_{j \geq n} \frac{|S_j|}{j} > \epsilon\right) \rightarrow 0 \forall \epsilon > 0:$$

To consider its rate of convergence, we shall use the condition that for some  $p \in (1; 2]$  and  $q \in [1; \infty)$  with  $q > (p - 1) = (p - 1)$ ,

$$\sup_{n \geq 1} \|\underline{m}_n(\cdot; n)\|_q < \infty; \quad (C3)$$

where  $\underline{m}_n(\cdot; n) = \frac{1}{n} \sum_{j=1}^n E[|X_j| | \mathcal{F}_{j-1}]$ . Remark that (C3) holds evidently if for some constant  $C > 0$  and all  $j \geq 1$ ,

$$E[|X_j| | \mathcal{F}_{j-1}] \leq C \text{ a.s.} \quad (C4)$$

**Theorem 3** Let  $p > 1$  and  $\psi \geq 0$  be slowly varying at  $\infty$ . Under (C3) or (C4), the following assertions are equivalent:

$$\sum_{n=1}^{\infty} n^{p-2} \psi(n) \sum_{j=1}^n P\{|X_j| > n^\alpha\} < \infty \quad \forall \alpha > 0; \quad (7)$$

$$\sum_{n=1}^{\infty} n^{p-2} \psi(n) P\left\{\sup_{1 \leq j \leq n} |S_j| > n^\alpha\right\} < \infty \quad \forall \alpha > 0; \quad (8)$$

$$\sum_{n=1}^{\infty} n^{p-2} \psi(n) P\{|S_n| > n^\alpha\} < \infty \quad \forall \alpha > 0; \quad (9)$$

$$\sum_{n=1}^{\infty} n^{p-2} \psi(n) P\left\{\sup_{j \geq n} \frac{|S_j|}{j} > \alpha\right\} < \infty \quad \forall \alpha > 0; \quad (10)$$

**Remark.** If  $X_j$  are identically distributed, then (7) is equivalent to the moment condition

$$E|X_1|^{\rho}(|X_1|) < \infty:$$

So Theorem 3 is an extension of the result of Baum and Katz (1965). When  $\rho$  is a constant, it was proved by Alsmeyer (1991).

**Theorem 4** Let  $p > 1$  and  $\nu \geq 0$  be slowly varying at  $\infty$ . Under (C3) or (C4), the following assertions are equivalent:

$$n^{p-1} \nu(n) \sum_{j=1}^n P\{|X_j| > n^\nu\} = o(1) \quad (\text{resp: } O(1)) \quad \forall \nu > 0; \quad (11)$$

$$n^{p-1} \nu(n) P\left\{ \sup_{1 \leq j \leq n} |S_j| > n^\nu \right\} = o(1) \quad (\text{resp: } O(1)) \quad \forall \nu > 0; \quad (12)$$

$$n^{p-1} \nu(n) P\{|S_n| > n^\nu\} = o(1) \quad (\text{resp: } O(1)) \quad \forall \nu > 0: \quad (13)$$

$$n^{p-1} \nu(n) P\left\{ \sup_{j \geq n} \frac{|S_j|}{j} > \nu \right\} = o(1) \quad (\text{resp: } O(1)) \quad \forall \nu > 0: \quad (14)$$

#### 4. Applications to sums of weighted random variables.

Example: Cesàro summation for martingale differences.

For  $a > -1$ , let  $A_0^a = 1$  and

$$A_n^a = \frac{(a+1)(a+2)\cdots(a+n)}{n!}; \quad n \geq 1:$$

Then  $A_n^a \sim \frac{n^a}{\Gamma(a+1)}$  as  $n \rightarrow \infty$ ; and  $\frac{1}{A_n^a} \sum_{j=0}^n A_{n-j}^{a-1} = 1$ . We consider convergence rates of

$$\frac{\sum_{j=0}^n A_{n-j}^{a-1} X_j}{A_n^a};$$

where  $\{(X_j; \mathcal{F}_j); j \geq 0\}$  are martingale differences that are identically distributed.

For simplicity, suppose that for some  $\alpha \in (1; 2]$ ;  $C > 0$  and all  $j \geq 1$ ,

$$\mathbb{E} [ |X_j| \mid \mathcal{F}_{j-1} ] \leq C \alpha^j \quad (15)$$

**Theorem 5.** Let  $\{(X_j; \mathcal{F}_j); j \geq 0\}$  be identically distributed martingale differences satisfying (15). Let  $p \geq 1$ , and assume that

$$\left\{ \begin{array}{ll} E|X_1|^{\frac{p-1}{a+1}} < \infty & \text{if } 0 < a < 1 - \frac{1}{p}; \\ E|X_1|^p \log(e \vee |X_1|) < \infty & \text{if } a = 1 - \frac{1}{p}; \\ E|X_1|^p < \infty & \text{if } 1 - \frac{1}{p} < a \leq 1: \end{array} \right. \quad (16)$$

Then

$$\sum_{n=1}^{\infty} n^{p-2} P\left\{ \left| \sum_{j=0}^n A_{n-j}^{a-1} X_j \right| > A_n^a \right\} < \infty \text{ for all } \epsilon > 0: \quad (17)$$

**Remark:** in the independent case, the result is due to Gut (1993).

## 5. Proofs of main results

The proofs are based on some maximal inequalities for martingales.



A. Relation between

$$P\left(\max_{1 \leq j \leq n} |X_j| > \epsilon\right) \text{ and } P\left(\max_{1 \leq j \leq n} |S_j| > \epsilon\right)$$

for martingale differences  $(X_j)$ :

**Lemma A** Let  $\{(X_j; \mathcal{F}_j); 1 \leq j \leq n\}$  be a finite sequence of martingale differences. Then for any  $\epsilon > 0$ ;  $\epsilon \in (1; 2]$ ;  $q \geq 1$ , and  $L \in \mathbb{N}$ ,

$$\begin{aligned}
 P\left\{\max_{1 \leq j \leq n} |X_j| > 2\epsilon\right\} &\leq P\left\{\max_{1 \leq j \leq n} |S_j| > \epsilon\right\} \\
 &\leq P\left\{\max_{1 \leq j \leq n} |X_j| > \frac{\epsilon}{4(L+1)}\right\} \\
 &\quad + C \epsilon^{-\frac{q(L+1)}{q+L}} (E m_n^q(\cdot))^{1+\frac{L}{q+L}}; \quad (18)
 \end{aligned}$$

where  $C = C(\cdot; q; L) > 0$  is a constant depending only on  $\cdot$ ;  $q$  and  $L$ ,

$$m_n(\cdot) = \sum_{j=1}^n E[|X_j| | \mathcal{F}_{j-1}]:$$

## B. Relation between

$$P\left(\max_{1 \leq j \leq n} X_j > \epsilon\right) \text{ and } \sum_{1 \leq j \leq n} P(X_j > \epsilon)$$

for adapted sequences  $(X_j)$ :

**Lemma B** Let  $\{(X_j; \mathcal{F}_j); 1 \leq j \leq n\}$  be an adapted sequence of r.v. Then for  $\epsilon > 0$ ;  $\delta > 0$  and  $q \geq 1$ ,

$$\begin{aligned} P\left\{\max_{1 \leq j \leq n} X_j > \epsilon\right\} &\leq \sum_{j=1}^n P\{X_j > \epsilon\} \\ &\leq (1 + \epsilon^{-q}) P\left\{\max_{1 \leq j \leq n} X_j > \epsilon\right\} + \epsilon^{-q} E m_n^q(\epsilon); \end{aligned}$$

where  $m_n(\epsilon) = \sum_{j=1}^n E[|X_j| | \mathcal{F}_{j-1}]$ .

## C. Relation between

$$P\left(\max_{1 \leq j \leq n} |S_j| > \epsilon\right) \text{ and } P(|S_n| > \epsilon)$$

for martingale differences  $(X_j)$ :

**Lemma C** Let  $\{(X_j; \mathcal{F}_j); 0 \leq j \leq n\}$  be a finite sequence of martingale differences. Then for  $\epsilon > 0$ ;  $\epsilon \in (1; 2]$  and  $q \geq 1$ ,

$$P\left\{\max_{1 \leq j \leq n} |S_j| > \epsilon\right\} \leq 2P\left\{|S_n| > \frac{\epsilon}{2}\right\} + \epsilon^{-q} 2^{q(n+1)} C^q(\epsilon) E m_n^q(\epsilon);$$

where  $m_n(\epsilon) = \sum_{j=1}^n E[|X_j| | \mathcal{F}_{j-1}]$ ,

$$C(\epsilon) = \left(18 \epsilon^{3-2} = (\epsilon - 1)^{1-2}\right).$$

---

Thank you!

Quansheng.Liu@univ-ubs.fr

---